

Ring sequence decomposition of an accretion disk: the viscoresistive approach

R. Benini¹G. Montani^{1,2,3}J. Petitta¹

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¹ Department of Physics - “Sapienza” Università di Roma,
Piazzale A. Moro, 5 (00185), Rome, Italy

² ENEA - C.R. Frascati (Department UTFUS-MAG)
Via Enrico Fermi, 45 (00044), Frascati (Rome), Italy

³ Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Roma 1

Abstract

We analyze a two dimensional viscoresistive magnetohydrodynamical (MHD) model for a thin accretion disk which reconciles the crystalline structure outlined in [Coppi(2005), Coppi and Rousseau(2006)] with real microscopic and macroscopic features of astrophysical accreting systems. In particular, we consider small dissipative effects (viscosity and resistivity, characterized by a magnetic Prandtl number of order unity), poloidal matter fluxes and a toroidal component of the magnetic field. These new ingredients allow us to set up the full equilibrium profile including the azimuthal component of the momentum conservation equation and the electron force balance relation. These two additional equations, which were identically satisfied in the original model, permit us to deal with non zero radial and vertical matter fluxes, and the solution we construct for the global equilibrium system provides a full description of the radial and vertical dependence within the plasma disk. The main issue of our analysis is outlining a modulation of the matter distribution in the disk which corresponds to the formation of a ring-like sequence, here associated with a corresponding radial oscillation of the matter flux.

compact object. Starting from the original idea of Shakura [Shakura(1973)] a wide number of results (see for instance [Pringle and Rees(1972)], [Shakura and Sunyaev(1988)], or [Lynden-Bell and Pringle(1974), Lynden-Bell(1996)]) allowed to achieve the firm understanding that, within a disk, the angular momentum transport is driven by dissipative effects (for a detailed review see [Bisnovatyi-Kogan and Lovelace(2001)]). Since the nature of such viscosity and resistivity of the plasma cannot be recovered on a microscopical level (such systems are mainly collisionless), then the origin of non-ideal features must rely on the turbulent behaviour of the plasma once linear instabilities are triggered. The most successful paradigm to explain the onset of the turbulence is the so called Magneto-Rotational Instability, discovered by Velikhov [Velikhov(1959)] and Chandrasekhar [Chandrasekhar(1960)], and recently fully developed by Balbus and collaborators [Balbus and Hawley(1991)] (for a review see also [Balbus and Hawley(1998)]). However, such instability, which can be triggered by an arbitrary small magnetic field, is nonetheless suppressed when the sound speed is smaller than the Alfvén velocity, i.e. when the β parameter of the plasma is smaller than one. Since this condition can take place in sufficiently cold and magnetized plasma disks, a challenging puzzle arises when dealing with the accretion mechanism in structures exhibiting such regimes.

1 Introduction

The relevance of the accretion processes in understanding the emission properties of many astrophysical sources like Gamma Ray Bursts [Piran(1999)] and Active Galactic Nuclei [Krolik(1999)], led to a significant effort over the last four decades in order to construct a standard model for the profile of an accreting disk over a com-

Two interesting proposals for alternative scenarios were proposed in [Bisnovatyi-Kogan and Lovelace(2000)] and [Coppi and Coppi(2001)]. In particular, B. Coppi and coworkers developed over the last ten years a line of research devoted to construct a different paradigm [Coppi(2009)], mainly based on the implementation on

the astrophysical setting of some basic features observed in laboratory plasma, like ballooning mode instabilities [Coppi and Keyes(2003)]. More than solving the problem of accretion for an ideal plasma disk, such efforts had the main merit to outline the existence of regimes where the disk profile can fragment into a sequence of rings. This situation takes place as soon as the Lorentz force, produced by the plasma backreaction in the magnetic field of the central object, plays a significant role in the vertical confinement. This result, however, was obtained in a local description of a purely rotating disk, and disregarding any dissipative effect (see also [Montani and Carlevaro(2010)] for the jet formation in this scheme). Despite being rather small, such dissipation features are present in any real astrophysical system, and the possibility to reconcile the ring sequence with a realistic equilibrium configuration is the main task addressed by the present work.

We consider the full MHD equilibrium including, with respect to the original model, the azimuthal momentum conservation equation and the electron force balance relation. For a purely ideal and rotating plasma, these equations are identically satisfied, but in a real system (here we consider non zero viscosity and resistivity coefficient, associated to a magnetic Prandtl number of order unity) they are crucial for fixing the poloidal matter fluxes and the toroidal magnetic field. We provide a solution for the global configuration system which has the merit to demonstrate that the existence of the ring sequence within the disk is compatible with non zero dissipative effects, and, over all, with the presence of a non zero radial matter flux. Nonetheless, the smallness of the additional terms we include in our treatment, allows us to neglect the poloidal velocity and the toroidal magnetic field in the radial and vertical configuration, which therefore retain the same form as in [Coppi(2005), Coppi and Rousseau(2006)]. We emphasize that dealing with only small corrections is a constraint imposed by the corotation theorem [Ferraro(1937)]. In fact, as shown in the last section, up to the leading order, it is still possible to preserve the relation between the angular velocity of the disk ω and the magnetic surface function ψ as far as the additional contributions due to the poloidal velocity, the toroidal magnetic field and the dissipative effects, are sufficiently small with respect to the basic model quantities as in [Coppi(2005), Coppi and Rousseau(2006)]. A significant improvement that our analysis is able to provide, is fixing the vertical dependence of the disk profile as a consequence of a compatibility equation coming out from the azimuthal and the electron force balance system. The local model we provide can be regarded as the starting point for the study of the settlement of linear instabilities in the paradigm of the crystalline structure.

As outlined in [Montani and Benini(2011)], the crystalline morphology of the magnetic field, as well as the ring sequence profile, have a characteristic scale which turns out to be microscopical on an astrophysical setting (indeed the

typical length of the perturbations results to be of the order of some meters). This issue suggests that, more than for its stationary appearance, such a morphology concerns the stability structure of the disk, and only after a full scheme of perturbation analysis is developed, a precise nature of the configuration can be settled down. In this respect, our treatment stands as the natural and realistic framework where to study the stability of the plasma disk.

2 Two-Dimensional MHD Model for an Accretion Disk

We will now set up the basic 2D MHD configuration equations for an axially-symmetric disk around a compact (few Solar masses) and strongly magnetized (a dipole-like field of about 10^{12} gauss) object, i.e. a typical pulsar source. In such a system, we neglect the self-gravity of the disk, so that the gravitational potential χ , in cylindrical coordinates (r, z, ϕ) , has the form

$$\chi(r, z) = \frac{GM_*}{\sqrt{r^2 + z^2}}, \quad (1)$$

where M_* is the mass of the central body, and G the gravitational constant. It is worth noting that the axial symmetry prevents any dependence on the azimuthal angle ϕ of all the quantities involved in the problem. The continuity equation associated to the mass conservation, i.e.

$$\vec{\nabla} \cdot (\epsilon \vec{v}) = 0, \quad (2)$$

admits the following explicit solution

$$\epsilon \vec{v} = \frac{1}{r} \vec{\nabla} \Theta \wedge \hat{e}_\phi + \epsilon \omega r \hat{e}_\phi. \quad (3)$$

Here ϵ denotes the mass density, \vec{v} the velocity field of the disk, and $\Theta(r, z)$ is a function which is odd with respect to the z variable, in order to deal with a non-zero accretion rate \dot{M}_d . In fact, if H is the half-depth of the disk, \dot{M}_d results to be given by [Bisnovatyi-Kogan and Lovelace(2001), Montani and Benini(2011)]

$$\dot{M}_d = -2\pi r \int_{-H}^H \epsilon v_r dz = 4\pi \Theta(r, H) > 0. \quad (4)$$

The dynamics of such a stationary, viscoresistive system is then described by the following equation [Bisnovatyi-Kogan and Lovelace(2001)]

$$\begin{aligned} \epsilon (\vec{v} \cdot \vec{\nabla}) \vec{v} = & -\vec{\nabla} \left(p + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} (\vec{B} \cdot \vec{\nabla}) \vec{B} + \\ & + \epsilon \vec{\nabla} \chi + \nabla_k \left[\mathbb{D} \left(\nabla_k \vec{v} + \nabla \vec{v}_k - \frac{2}{3} (\vec{\nabla} \cdot \vec{v}) \right) \right] \mathbb{I}, \end{aligned} \quad (5)$$

\mathbb{D} being the viscosity and \mathbb{I} the unit tensor; the magnetic field \vec{B} , because of the axial symmetry, can be expressed

through the magnetic flux function $\psi = \psi(r, z^2)$ in the general form

$$\vec{B} = \frac{1}{r} \vec{\nabla} \psi \wedge \hat{e}_\phi + \frac{I}{r} \hat{e}_\phi, \quad (6)$$

where $I = I(\psi, z)$ is a generic function of ψ characterizing the toroidal component of the field. The similarity of the magnetic field and matter flux structure, is due to their common divergence-less nature. A further assumption in our derivation is that the angular frequency of the disk ω has to be a function of ψ , i.e.

$$\omega = \omega(\psi). \quad (7)$$

This result has been firstly demonstrated by Ferraro in [Ferraro(1937)] under the hypotheses of an ideal, non resistive, axisymmetric rotating plasma. It can be shown (see the detailed proof discussed in the last section) that this result holds even in the presence of small resistivity and viscosity.

2.1 Local perturbative analysis

In the present analysis, we are interested in the behavior of the configuration around a fixed, fiducial value r_0 of the radial variable r . In order to investigate the effects induced on the disk profile by the electromagnetic reaction of the plasma, we split the mass density and the pressure contributions in the zeroth (barred) and the first (hatted) orders components, i.e.

$$\epsilon = \bar{\epsilon}(r_0, z^2) + \hat{\epsilon}, \quad p = \bar{p}(r_0, z^2) + \hat{p}, \quad (8)$$

respectively. The same way, we express the magnetic surface function as the sum of a term ψ_0 , describing the field of the star, plus a correction ψ_1

$$\psi = \psi_0(r_0) + \psi_1(r_0, r - r_0, z^2), \quad \psi_1 \ll \psi_0. \quad (9)$$

The quantities $\hat{\epsilon}$, \hat{p} and ψ_1 describe the change induced by the currents which rise within the disk. In general, these corrections are small in amplitude but with a very short scale of variation. Having this scenario in mind, we address the so-called "drift ordering" for the behavior of the gradient amplitude, i.e. the first order gradients of the perturbations are of zeroth-order, while the second order ones dominate. Accordingly, the profile of the toroidal currents rising in the disk, has the form

$$J_\phi = \frac{c}{4\pi} \vec{\nabla} \wedge \vec{B} \simeq \frac{-c}{4\pi r_0} (\partial_r^2 \psi_1 + \partial_z^2 \psi_1) = \frac{-c}{4\pi r_0} \Delta \psi_1. \quad (10)$$

On the other hand, the azimuthal component of the Lorentz force is related to the existence of the function $I(\psi, z)$ and results to be equal to

$$F_\phi \simeq \frac{1}{4\pi r_0^2} (\partial_z I \partial_r \psi - \partial_r I \partial_z \psi). \quad (11)$$

As a consequence of the corotation theorem, in the present split scheme, we can decompose ω in the sum of the Keplerian term ω_K and a constant ω'_0 , respectively given by

$$\omega = \omega_K + \omega'_0 \psi_1 \quad (12)$$

$$\omega_K \equiv \omega(\psi_0) = \sqrt{\frac{GM_*}{r_0^3}}, \quad \omega'_0 \equiv \left. \frac{d\omega}{d\psi} \right|_{\psi_0} \quad (13)$$

This form for ω holds locally, as far as $(r - r_0)$ remains a sufficiently small quantity, so that the dominant deviation from the Keplerian contribution is due to ψ_1 only.

2.2 Vertical dynamics

As developed in [Coppi(2005), Coppi and Rousseau(2006)], we can distinguish the fluid components from the electromagnetic back-reaction. The vertical component of eq.(5) implies that, at the zeroth order, the background profile is determined by the pure thermostatic equilibrium standing in the disk when the vertical gravity (i.e. the Keplerian rotation) is sufficiently high to provide a confined thin configuration

$$D(z^2) \equiv \frac{\bar{\epsilon}}{\epsilon_0(r_0)} = \exp\left(-\frac{z^2}{H^2}\right) \quad (14)$$

$$\epsilon_0(r_0) \equiv \epsilon(r_0, 0), \quad H^2 \equiv \frac{4K_B \bar{T}}{m\omega_K^2},$$

Here we have assumed that the background obeys an isothermal relation, and that the temperature T may be then represented as

$$2K_B T \equiv m \frac{p}{\epsilon} = m \frac{\bar{p} + \hat{p}}{\bar{\epsilon} + \hat{\epsilon}} \equiv 2K_B (\bar{T} + \hat{T}), \quad (15)$$

K_B being the Boltzmann constant and m the ion mass. On the other hand, at the first order, the vertical equilibrium is described by the following equation

$$\partial_z \hat{p} + \omega_K^2 z \hat{\epsilon} - \frac{1}{4\pi r_0^2} \Delta \psi_1 \partial_z \psi_1 = 0. \quad (16)$$

2.3 Radial equilibrium

The radial component of eq.(5) fixes the equilibrium features of the rotating layers of the disk, and can be decomposed into the dominant character of the Keplerian angular velocity, plus an equation describing the behavior of the deviation, i.e.

$$2\omega_K r_0 (\bar{\epsilon} + \hat{\epsilon}) \omega'_0 \psi_1 + \frac{1}{4\pi r_0^2} \Delta \psi_1 \partial_r \psi_0 = \partial_r \left[\hat{p} + \frac{1}{8\pi r_0^2} (\partial_r \psi_1)^2 \right] + \frac{1}{4\pi r_0^2} \partial_r \psi_1 \partial_z^2 \psi_1. \quad (17)$$

Here we neglected the presence of the poloidal currents, associated with the azimuthal component of the magnetic field, as soon as their contribution results to be of higher order.

2.4 Azimuthal equation

By inserting eq.(12) into the ϕ -component of eq.(5), we easily obtain the relation governing the azimuthal equilibrium; more precisely we get

$$\begin{aligned} \epsilon v_r \partial_r \psi + \epsilon v_z \partial_z \psi + 2\epsilon v_r \frac{\omega}{\omega'_0} &= \\ = \frac{1}{r^2 \omega'_0} \partial_r (\mathbb{D} r^3 \omega'_0 \partial_r \psi) + \frac{1}{\omega'_0} \partial_z (\mathbb{D} r \omega'_0 \partial_z \psi) + \frac{F_\phi}{\omega'_0}. \end{aligned} \quad (18)$$

By virtue of the magnetic field form (6), of the Lorentz force (11), the azimuthal equation stands at r_0 as

$$\begin{aligned} \epsilon r_0^2 (v_r B_z - v_z B_r) + 2\epsilon v_r \frac{\omega_K}{\omega'_0} &= r_0 \mathbb{D}_0 \Delta \psi_1 + \\ + \frac{1}{4\pi r_0^2 \omega'_0} [\partial_z I (\partial_{r_0} \psi_0 + \partial_r \psi_1) - \partial_r I \partial_z \psi_1], \end{aligned} \quad (19)$$

where $\mathbb{D}_0 = \mathbb{D}(r_0)$ and we neglected the z -derivatives of the viscosity coefficient in view of the hierarchy of the equations [Montani and Benini(2011)]. This relation accounts for the angular momentum transport across the disk, by relying on the presence of a viscous feature of the differential rotation. Furthermore, this equation establishes a link between the radial and vertical velocity fields with the poloidal currents present in the configuration.

A complete analysis of the dynamics is, however, still not possible. The system given by Eqs.(16), (17) and (19) is not closed, and we have to impose some other condition in order to describe the poloidal component of the magnetic field I and the function Θ . This task will be accomplished in the next Section when the balance of the forces acting on the electrons will be discussed.

3 The electron force balance equation

In the presence of a non-zero resistivity coefficient η , the equation accounting for the electron force balance reads as

$$\vec{E} + \frac{\vec{v}}{c} \wedge \vec{B} = \eta \vec{J}. \quad (20)$$

Since the contribution of the resistive term is expected to be relatively small, it turns out as appreciable only in the azimuthal component of the equation above. Thus, the balance of the Lorentz force has dominant radial and vertical components, providing the electric field in the form predicted by the corotation theorem, i.e.

$$\vec{E} = -\frac{\vec{v}}{c} \wedge \vec{B} = -\frac{\omega}{c} (\partial_r \psi \vec{e}_r + \partial_z \psi \vec{e}_z). \quad (21)$$

Since the axial symmetry requires $E_\phi \equiv 0$, the ϕ -component of the equation above takes the form

$$v_z B_r - v_r B_z = c\eta J_\phi, \quad (22)$$

and, taking into account eq.(10), it reduces to

$$v_r B_z - v_z B_r = \frac{\eta c^2}{4\pi r_0} \Delta \psi_1. \quad (23)$$

This relation, together with the azimuthal equilibrium equation (19), provides a system for the two unknowns Θ and I , when ψ_1 and ϵ are given by the vertical (16) and radial (17) equilibria. More precisely, substituting eq.(23) in eq.(19), we get

$$\begin{aligned} 2\epsilon v_r \frac{\omega_K}{\omega'_0} &= r_0 \left(\mathbb{D}_0 - \frac{c^2 \eta \epsilon}{4\pi} \right) \Delta \psi_1 + \\ + \frac{1}{4\pi r_0^2 \omega'_0} [\partial_z I (\partial_{r_0} \psi_0 + \partial_r \psi_1) - \partial_r I \partial_z \psi_1]. \end{aligned} \quad (24)$$

4 Non-linear Configuration

In order to analyze the dynamics, we need to state some relation between the viscosity \mathbb{D} and the resistivity η of the plasma. As already done in [Montani and Benini(2011)], we assume that the magnetic Prandtl number of the plasma is of order one. This in turns implies that we can take the following relation

$$\eta = \frac{4\pi \mathbb{D}_0}{c^2 (\bar{\epsilon} + \hat{\epsilon})}. \quad (25)$$

Now, inserting eq.(25) into eq.(24), we can get a direct relation between the radial velocity v_r , that accounts for the accretion on the central object, and the Lorentz force F_ϕ ; more precisely, we get

$$v_r = \frac{1}{2\omega_K (\bar{\epsilon} + \hat{\epsilon})} F_\phi. \quad (26)$$

Substituting v_r by this equation into the electron force balance (23), we get the following expression for v_z

$$v_z = \frac{1}{\partial_z \psi_1 (\bar{\epsilon} + \hat{\epsilon})} \left[\mathbb{D}_0 \Delta \psi_1 - (\partial_{r_0} \psi_0 + \partial_r \psi_1) \frac{F_\phi}{2\omega_K} \right]. \quad (27)$$

In terms of the function Θ , the expressions above for v_r (26) and v_z (27) stands as

$$\partial_z \Theta = -\frac{r_0}{2\omega_K} F_\phi, \quad (28)$$

$$\partial_r \Theta = \frac{r_0}{\partial_z \psi_1} \left[\mathbb{D}_0 \Delta \psi_1 - \frac{1}{2\omega_K} (\partial_{r_0} \psi_0 + \partial_r \psi_1) F_\phi \right]. \quad (29)$$

The solution of the first equation can be taken as

$$\Theta = -\frac{r_0}{2\omega_K} \int dz (F_\phi), \quad (30)$$

which, substituted in the second, yields an integro-differential relation for F_ϕ , i.e.

$$\partial_z \psi_1 \int dz (\partial_r F_\phi) = (\partial_{r_0} \psi_0 + \partial_r \psi_1) F_\phi - 2\omega_K \mathbb{D}_0 \Delta \psi_1. \quad (31)$$

Once the behavior of ψ_1 is provided, from the equation above we can determine the form of $F_\phi(r, z)$ and eventually calculate the ϕ -component $I(\psi, z)$ of the magnetic field.

4.1 Dimensionless variables

Let us define the dimensionless functions Y , \hat{D} and \hat{P} , in place of ψ_1 , $\hat{\epsilon}$ and \hat{p} , i.e.

$$Y \equiv \frac{k_0 \psi_1}{\partial_{r_0} \psi_0}, \quad \hat{D} \equiv \beta \frac{\hat{\epsilon}}{\epsilon_0}, \quad \hat{P} \equiv \beta \frac{\hat{p}}{p_0}, \quad (32)$$

where

$$\beta \equiv 8\pi \frac{p_0}{B_{0z}^2} = \frac{1}{3\epsilon_z^2} \equiv k_0^2 \frac{H^2}{3}, \quad (33a)$$

$$p_0 \equiv 2K_B \bar{T} \epsilon_0, \quad k_0 \equiv 3 \frac{\omega_K^2}{v_A^2}, \quad v_A^2 \equiv \frac{B_{z0}^2}{4\pi \epsilon_0}, \quad (33b)$$

and $B_{z0} = B_z(r, z=0) = \partial_{r_0} \psi_0 / r_0$. We introduced the fundamental wavenumber k_0 that quantifies the typical length scale of the perturbation along the radial directions. Finally, we rescale the radial variable $x \equiv k_0(r-r_0)$, and the vertical one $u \equiv z/L_z$ assuming the fundamental length in the vertical direction to be $L_z \equiv \sqrt{\epsilon_z} H$.

By these definitions, the vertical (16) and the radial equilibria (17) can be restated as

$$\partial_{u^2} \hat{P} + \epsilon_z \hat{D} + 2\Delta_\epsilon Y \partial_{u^2} Y = 0, \quad (34a)$$

$$\left(D + \frac{1}{\beta} \hat{D}\right) Y + \Delta_\epsilon Y (1 + \partial_x Y) + \frac{1}{2} \partial_x \hat{P} = 0, \quad (34b)$$

where

$$\Delta_\epsilon Y \equiv \partial_x^2 Y + \epsilon_z \partial_u^2 Y. \quad (35)$$

The two equations above provide a coupled system for \hat{P} and Y once the quantities D and \hat{D} are assigned; hence we are able to determine the disk configuration induced by the toroidal currents. Finally, eq.(31) takes the form

$$\partial_u Y \int du (\partial_x A_\phi) = -\Delta_\epsilon Y + (1 + \partial_x Y) A_\phi, \quad (36)$$

$A_\phi \equiv F_\phi / 2\omega_K \mathbb{D}_0 k_0$ being a dimensionless function.

5 The ring sequence

Here we will derive a solution to the system (34) in the case $\epsilon_z \neq 0$, showing how the formation of the crystalline structure for the magnetic field and the decomposition of the disk in a ring sequence can happen also in different regimes from those addressed in [Coppi(2005), Coppi and Rousseau(2006), Montani and Benini(2011)]. Let us look for a particular solution of the form

$$Y(x, u^2) = F(u^2) \sin(ax), \quad (37a)$$

$$\hat{D}(x, u^2) = \Gamma(u^2) e^{b \cos(ax)}, \quad (37b)$$

$$\hat{P}(x, u^2) = M(u^2) \sin^2(ax) + N(u^2) e^{b \cos(ax)}, \quad (37c)$$

where a, b are some parameters to be fixed. We assume to be very close to the equatorial plane, in order to approximate the function \bar{D} to a constant

$$|u| \ll \frac{1}{\sqrt{\epsilon_z}} \Rightarrow \bar{D}(u^2) = \exp(-\epsilon_z u^2) \simeq 1. \quad (38)$$

Inserting eqs.(37) in (34), the following system can be easily obtained

$$F'' - B^2 F = 0, \quad M = F^2, \quad (39a)$$

$$\frac{6\epsilon_z F}{ab} \dot{N} + N = 0, \quad \Gamma = -\frac{\dot{N}}{\epsilon_z}, \quad (39b)$$

where $()' \equiv d/du$, $\dot{()} \equiv d/du^2$, and B a positive constant

$$B^2 = \frac{a^2 - 1}{\epsilon_z} > 0 \Rightarrow a^2 > 1. \quad (40)$$

The solution to system (39) reads as

$$F = A e^{-B\sqrt{u^2}}, \quad (41a)$$

$$M = A^2 e^{-2B\sqrt{u^2}}, \quad (41b)$$

$$N = C \exp\left(\frac{ab}{3A(a^2-1)} e^{B\sqrt{u^2}} (1 - B\sqrt{u^2})\right), \quad (41c)$$

$$\Gamma = \frac{abC}{6\epsilon_z^2 A} \exp\left[\frac{ab(1 - B\sqrt{u^2}) e^{B\sqrt{u^2}}}{3A(a^2-1)} + B\sqrt{u^2}\right], \quad (41d)$$

where A and C are integration constants. The finiteness of the solutions, leads us to require

$$\frac{ab}{A} > 0. \quad (42)$$

Finally, the total mass density assumes the following form

$$\begin{aligned} D(x, u^2) &= \bar{D} + \frac{1}{\beta} \hat{D} = \\ &= 1 + \frac{abC}{2A} \exp\left[-\frac{abe^{B\sqrt{u^2}}}{3A(a^2-1)} (B\sqrt{u^2} - 1)\right] \times \\ &\times \exp(B\sqrt{u^2} + b \cos(ax)), \end{aligned} \quad (43)$$

which results to be positive definite if the following condition holds:

$$\left|\frac{abC}{2A}\right| \exp\left(-\frac{ab}{3A(a^2-1)} + |b|\right) < 1. \quad (44)$$

The formation of the ring sequence can be seen in Figs.(1) and (2) representing the density (43) for a particular choice of the model parameters.

It is worth noting that solutions (41) satisfy also eq.(36) as soon as we fix the form of the Lorentz force as the following

$$A_\phi(x, u^2) = -F(u^2) \sin(ax). \quad (45)$$

Finally, from eq.(26), we obtain a non-zero radial velocity

$$v_r = -\frac{A \mathbb{D}_0 k_0}{\epsilon_0} \frac{e^{-B\sqrt{u^2}} \sin(ax)}{D}. \quad (46)$$

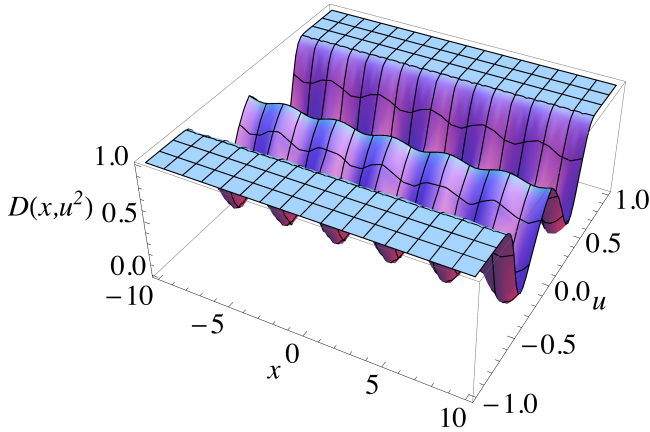


Figure 1: 3D-plot for the total density $D(x, u^2)$ (43), with parameters chosen as $A = 1, C = -1, a = -\sqrt{3}, b = -\frac{1}{6}, \varepsilon_z = 0.05, B = \sqrt{40}$. The density profile exhibits an modulation along the radial direction, while in the vertical direction two minima appear.

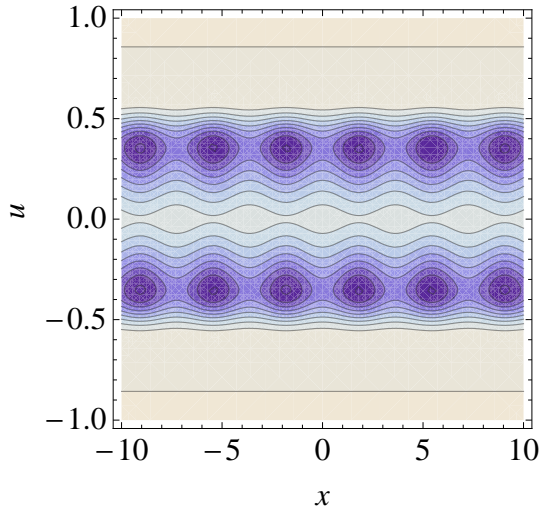


Figure 2: Level lines for the total density $D(x, u^2)$ (43), with parameters chosen as in fig.(1). We note that the darkest zones are the minima of density, and the ring sequence appears in the equatorial plane.

6 The corotation theorem for a weakly resistive and viscous plasma

In the following, we will show how the corotation theorem by Ferraro [Ferraro(1937)] holds under the hypothesis of stationarity up to the second order in small dissipative effects, poloidal velocity and toroidal component of the magnetic field.

Let us consider the Maxwell equations

$$\eta \vec{J} = \vec{E} + \frac{1}{c} \vec{v} \wedge \vec{B}, \quad (47)$$

$$\vec{\nabla} \wedge \vec{B} = \frac{4\pi}{c} \vec{J}, \quad (48)$$

and combine them by eliminating \vec{J}

$$c\vec{E} = \frac{c^2}{4\pi} \eta \vec{\nabla} \wedge \vec{B} - \vec{v} \wedge \vec{B}. \quad (49)$$

In the case of a steady-state configuration, the vector \vec{E} is irrotational. So, as soon as the poloidal velocity and the toroidal magnetic component are small enough, and by using eq.(3) and eq.(6), eq.(49) reduces to

$$\vec{\nabla} \wedge \begin{pmatrix} \frac{4\pi}{c^2} \omega \partial_r \psi + \eta \frac{1}{r} \partial_z I \\ \frac{4\pi}{c^2} (v_r B_z - v_z B_r) + \eta \left[\frac{\partial_z^2 \psi}{r} + \partial_r \left(\frac{\partial_r \psi}{r} \right) \right] \\ \frac{4\pi}{c^2} \omega \partial_z \psi - \eta \frac{1}{r} \partial_r I \end{pmatrix} = 0. \quad (50)$$

The radial and vertical components of eq.(50), which are of first order in the small quantities, are identically satisfied as soon as one imposes the electron force balance (22). On the other hand, the azimuthal equation can be rewritten for convenience in the vector form

$$\frac{4\pi}{c^2} (\vec{\nabla} \omega \wedge \vec{\nabla} \psi) = \vec{\nabla} \wedge \left(\eta \frac{\vec{\nabla} I}{r} \wedge \hat{\phi} \right). \quad (51)$$

The corotation theorem, stating that $\vec{\nabla} \omega \wedge \vec{\nabla} \psi = 0$, can be recovered up to the first order because the right hand side of the last equation is of second order in η, I . Indeed, we have to require not only η and I to be small, but also that their gradients must retain the same behavior. It is possible to check that this is the case allowed in the solution derived above (see the crossmath of eq.(11) and eq.(45)).

7 Concluding Remarks

We analyze an axis-symmetric thin disk configuration whose structure is summarized by a mainly rotating plasma embedded in a magnetic field endowed with a small toroidal component. The consistence of the configuration scheme

requires that correspondingly small poloidal matter fluxes and dissipative effects must be included in the problem. This analysis, which generalizes the original approach by Coppi and collaborators, has the merit of providing a coherent description of the vertical and radial dependence of the plasma profile. Matter fluxes result to be necessary when speaking of a real accreting disk, while the toroidal component of the magnetic field is expected to be non zero when dealing with the magnetic configuration of an astrophysical source. The profile we obtained corresponds to have a magnetic Prandtl number of order unity (which is a typical value required to phenomenologically address the observations [Spruit(2010)]), and it offers the natural scenario where a perturbation scheme of the ring sequence profile can be implemented.

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